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Generalized Semilocal Theories and Higher Hopf Maps

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Abstract

In semilocal theories, the vacuum manifold is fibered in a non-trivial way by the action of the gauge group. Here we generalize the original semilocal theory (which was based on the Hopf bundle $S^3 \xrightarrow{S^1} S^2$) to realize the next Hopf bundle $S^7 \xrightarrow{S^3} S^4$, and its extensions $S^{2n+1} \xrightarrow{S^3} P^n$. The semilocal defects in this class of theories are classified by $\pi_3(S^3)$, and are interpreted as constrained instantons or generalized sphaleron configurations. We fail to find a field theoretic realization of the final Hopf bundle $S^{15} \xrightarrow{S^7} S^8$, but are able to construct other semilocal spaces realizing Stiefel bundles over Grassmanian spaces.

One of the most lively facets of the cosmology/particle physics interface has been the utilization of topological defects arising from spontaneously broken symmetries in field theoretical models for cosmological purposes [1]. Inflation was initially devised as a way of getting around the inevitable existence of magnetic monopoles in grand unified theories [2], while cosmic strings [3] or cosmic texture [4] may be of vital importance in understanding how large scale structure is formed. The observation of the effects of topological defects would give us a low energy window into the high energy physics regime.

It is sometimes assumed that the types of finite energy defects supported by a given field theory were classified solely by the various homotopy groups of the manifold of degenerate vacua $M = G/H$, where G is the original symmetry group and H the unbroken subgroup [1]. However, care must be exercised when only a subgroup of G is gauged, for in this case it is only the coset space of gauged symmetries G_l/H_l that is relevant [5]. Indeed, Vachaspati and Achúcarro [6] (VA) constructed a simple model exhibiting finite energy stringlike configurations despite the vacuum manifold having a trivial first homotopy group. However, these so-called *semilocal strings* (which are essentially Nielsen-Olesen vortices [7]) are stable only when the Higgs mass is smaller than that of the gauge boson [8, 9], from which we learn that the usual interpretation of the homotopic classification of defects fails for semilocal theories. Preskill [10] has emphasised that the surprising feature of semilocal strings is that they are unstable for sufficiently large Higgs masses, and has also extended the analysis to models which possess approximate global symmetries over and above the exact ones. These would arise through gauge boson loops reducing an accidental symmetry of the potential not possessed by the whole Lagrangian, and can act to stabilize otherwise unstable defects. These theories have other fascinating properties [11, 12] and may have important consequences for cosmology [9]. Thus they merit thorough investigation.

The original semilocal theory [6, 8, 9] is one of $(n + 1)$ complex scalar fields with a global $U(n + 1)$ symmetry, of which the abelian $U(1)$ is gauged (this was termed an extended Abelian Higgs (EAH) model in references [8, 9]). The unbroken global symmetry is $U(n)$, and H_l is trivial. Thus the vacuum manifold is $U(n+1)/U(n) \simeq S^{2n+1}$, which is fibred by the S^1 gauge orbits in a non-trivial way: the base space is P^n [8, 9]. At low momenta (lower than the masses of the Higgs and vector particles) this base space is the target space of a non-linear σ -model, and the coordinates of this space are the $2n$ Goldstone modes of the theory. The gauge coset space M_l is $G_l/H_l \simeq S^1$, showing that the theory has non-trivial π_1 and therefore the possibility of vortex configurations.

In this paper we extend the class of semilocal theories in several directions, but first we review the relation between the vacuum manifold of the $n = 1$ EAH model and the Hopf bundle $S^3 \xrightarrow{S^1} S^2$. This discussion then sets the stage for the main work of this paper, which is the extension of the original semilocal theory to a class of field theories whose first member provides a model for the higher Hopf bundle $S^7 \xrightarrow{S^3} S^4$. The low momentum dynamics of this theory will turn out to be that of a *quaternionic* P^n non-linear σ -model [13]. We will then elucidate the nature of the defect found in this theory. There are a couple of other possible generalizations. One of these is to exploit the fact that $P^n \simeq SU(n + 1)/[SU(n) \times U(1)]$ is one member of the set of complex Grassman manifolds, $G(n, m) \simeq SU(n + m)/S[U(n) \times U(m)]$, and recast the σ -model construction [15] into that of a spontaneously broken gauge theory. We can also do this for the real Grassman manifolds $G(n, m) = SO(n + m)/S[O(n) \times O(m)]$. However, the step up from quaternions to octonions does not seem to provide us with yet another class.

The semilocal string model of VA is based on the symmetry group $G = [SU(2)_g \times U(1)_l]/_2$, where the subscripts g, l indicate whether the symmetry is global or local, respectively. G is then broken down to $H = U(1)_g$ by a complex doublet Φ . The generator of H is a linear combination of the diagonal generator of $SU(2)_g$ and the $U(1)_l$

generator. This is essentially the scalar sector of the electroweak model in the limit that the $SU(2)_L$ gauge coupling g is set to zero, a fact which was used by Vachaspati to point out the possibility of classically stable vortex solutions in the Standard Model [16].

Let us now look for topological defects in this theory. This means that we are looking for static solutions of the field equations which are separated from the vacuum $|\Phi| = 0$ by field configurations whose energy diverges proportionally to the volume in the infinite volume limit. These solutions need not have finite energy themselves, but typically one requires that their energy diverges less fast than the volume. The energy functional in D spatial dimensions for static configurations is

$$\mathcal{E} = \int d^D x \left[\frac{1}{4} F_{ij} F_{ij} + |D_i \Phi|^2 + V(\Phi) \right] \quad (1)$$

where the Higgs potential is $V(\Phi) = \frac{1}{2} \lambda (\Phi^\dagger \Phi - \eta^2)^2$. The requirement that the energy diverge less fast than the volume R^D means that Φ must approach the manifold of zeroes of V , the vacuum manifold M , faster than $|x|^{-D}$ at infinity. M is given by $\Phi^\dagger \Phi = \eta^2$, which is easily seen to be S^3 when Φ is written in terms of the four real fields that are the real and imaginary parts of each component. Now both $\pi_1(S^3)$ and $\pi_2(S^3)$ are trivial, so we might not expect to have either cosmic strings or monopoles arising from this symmetry breaking. However, this conclusion is premature. Let us examine the covariant derivative contribution to the energy functional. Naively, this has a potential divergence going as R^{D-2} . However, if Φ is a gauge transformation at infinity, this term can be finite. Thus requiring finite energy of a field configuration necessarily means that Φ must lie on a gauge orbit, which has the topology of the coset space $M_l \subset M$. Thus if $\pi_{D-1}(M_l)$ is non-trivial, we should expect to find finite energy defects. Finally, the divergence of the contribution of the gauge field to the total energy depends on how fast F_{ij} tends to zero, which in turn depends on how fast the currents producing the field vanish. This is a dynamical rather than a topological question.

In the case at hand, $U(1)_l \rightarrow 1$, so that the gauge part of the vacuum manifold is isomorphic to S^1 . Every point on the S^3 defined by the constraint on the vacuum value of Φ lies on an S^1 defined by the action of $U(1)_l$ on S^3 (*i.e.* the action of G_l is effective on M). It costs no energy to move along one of these circles in field space. In 2 space dimensions we can therefore wrap the field at spatial infinity Φ_∞ around any of these S^1 's, giving rise to an infinite number of distinct field configurations. Extremizing in one of these sectors in field configuration space should give rise to a string solution. Now since S^3 is simply connected, it would be tempting to argue that these strings are deformable to the vacuum (Φ_∞ constant) simply by shrinking the loop defined by Φ_∞ to a point in M , and thus the strings will be unstable. The catch in this argument is that in order to perform this deformation Φ_∞ must leave the set of gauge orbits, so that at infinity it has a component which is a *global* $SU(2)_g$ transformation. Thus we pay a logarithmically divergent penalty in gradient energy. There is, then, an infinite energy barrier separating the vortex configurations from the vacuum.

The non-trivial embedding of the gauge vacuum manifold $S^1 = U(1)/1$ in the full vacuum manifold S^3 tells us that what we are dealing with is a fiber bundle structure where the total space is S^3 , the fiber S^1 and the base space is the coset of S^3 by the S^1 action, *i.e.* S^2 . This is nothing but one of the celebrated Hopf fibrations of spheres by spheres [17], sometimes termed the monopole bundle [18]. The fact that the space obtained by modding out by the action of S^1 on S^3 is S^2 also indicates that the theory supports global monopole configurations. These monopoles are at the end of the strings [9, 11].

In the general EAH model the symmetry group is $[SU(n+1)_g \times U(1)_l]/_{n+1}$, $n > 1$, which is spontaneously broken by a complex $(n+1)$ -plet to $U(n)_g \sim [SU(n)_g \times U(1)_g]/_n$ [8, 9]. The vacuum manifold in this case is S^{2n+1} , while the gauge orbits are again just S^1 . Thus, despite the fact that for $n > 1$, $\pi_1(S^{2n+1})$ is trivial, this theory will still

support semilocal strings. Furthermore, the low momentum limit of this theory (when mean momenta are much less than the scalar and vector masses) can be described by the dynamics of the P^n nonlinear σ -model. From a bundle point of view, the above construction just defines the following extension of monopole Hopf bundle: $S^{2n+1} \xrightarrow{S^1} P^n$.

The other way of constructing semilocal defects is to make use of the other Hopf bundles $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$ and find field theoretic models that realize them as in the monopole bundle case. What we will find is that only the first of these, often called the instanton bundle, lends itself to such a realization. This is essentially because S^3 is also the group $SU(2)$, while S^7 is not a group at all.

There are a variety of equivalent ways of approaching the construction of the field theoretic model of the instanton Hopf bundle. For example, it is well known [17] that the various Hopf bundles are intimately related to the various division algebras (reals, complex numbers, quaternions and octonions), with the monopole and instanton bundles being related to the complex numbers and quaternions, respectively. So let us try replacing the complex $(n+1)$ -vector Φ by $(n+1)$ quaternions $Q \equiv (q_1, \dots, q_{n+1})$. The group of symmetries that leave the norm $|\Phi|^2$ invariant is $U(n+1,)$. Analogously, the quaternionic scalar product is invariant under the group $U(n+1,)$ [19], which is the group of unitary rank $n+1$ matrices with quaternionic entries. This group is isomorphic to the symplectic group of rank $n+1$, $Sp(n+1)$ [19].

We may represent the components of Q by 2×2 matrices, using the Pauli matrices as a basis for the nonreal elements:

$$q_a = q_a^0 \mathbf{1} + q_a^A (i\sigma^A) \quad (2)$$

with q_a^0 and q_a^A ($A \in \{1, 2, 3\}$) real. The scalar product is then defined with the usual matrix trace:

$$(Q, Q) \equiv \frac{1}{2} \sum_1^{n+1} \text{tr}(q_a^\dagger q_a) = \sum_1^{n+1} [(q_a^0)^2 + (q_a^i)^2] \quad (3)$$

The action of an element M_{ab} of $U(n+1, Q)$ can be chosen to be by right multiplication, so that

$$q_a \rightarrow q'_a = q_b M_{ab} \quad (4)$$

But there is also another symmetry of the scalar product, which is $q_a \rightarrow q'_a = s q_a$, with $s^\dagger s = \mathbf{1}$. Because we are representing the quaternions by matrices, it is clear that this group of symmetries is $SU(2)$, or equivalently $Sp(1)$. The full symmetry group is not quite the direct product of the two groups, because $s = -\mathbf{1}$ has the same action as $M_{ab} = -\mathbf{1}\delta_{ab}$. Therefore, for this quaternionic extension we have

$$G = [Sp(n+1)_g \times Sp(1)_l]/_2 \quad (5)$$

In the original semilocal theory, the overall complex phase of the scalar field was gauged. Thus the quaternionic analogy is to gauge an overall quaternionic phase – the $Sp(1)$ subgroup which acts from the left. The covariant derivative is therefore

$$D_\mu Q = (\partial_\mu - g A_\mu) Q \quad (A_\mu = \tfrac{1}{2} i \sigma^A A_\mu^A) \quad (6)$$

We can now write down the Lagrangian of the model:

$$\mathcal{L} = -\tfrac{1}{2}(F_{\mu\nu}, F^{\mu\nu}) + (D_\mu Q, D^\mu Q) - \tfrac{1}{2}\lambda \left((Q, Q) - \eta^2 \right)^2 \quad (7)$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - q[A_\mu, A_\nu]$. The potential is stable to radiative corrections, since the whole Lagrangian has been designed to be invariant under the symmetry group G .

The vacuum manifold M of this theory is S^{4n+3} , because the potential constrains the sums of the squares of $4(n+1)$ real numbers to be constant. The gauge orbits are 3-spheres, for they preserve the magnitude of $(q_a^0)^2 + (q_a^i)^2$ for each a . The unbroken subgroup consists entirely of global symmetries and is $[Sp(n) \times Sp(1)]/_2$. This can be ascertained by choosing the vacuum expectation value to be $Q_0 = (\eta\mathbf{1}, 0, \dots, 0)^T$, which is always possible since we have the freedom to make global $U(n+1,)$ rotations without

changing any of the physics. The unbroken $Sp(n)$ subgroup acts on the lower n components of Q_0 , while the unbroken global $Sp(1)$ is the diagonal subgroup of $Sp(n+1) \times Sp(1)$ consisting of elements of the form

$$h = s^{-1} \delta_{ab} \otimes s \quad (8)$$

This is the quaternionic analogue of the unbroken global $U(1)$ in the complex field case.

We now demonstrate that the low momentum dynamics of this theory are those of a σ -model with target space P^n . First, we recall the definition of this class of quaternionic projective spaces [19]. The manifold P^n is constructed from ${}^{n+1}-\{0\}$ (the set of non-zero quaternionic $(n+1)$ -vectors Q) by identifying all elements which are equivalent by left multiplication: *i.e.*, $Q \equiv Q'$ if and only if there is a quaternion p such that $Q' = pQ$. This exactly parallels the construction of P^n . Equally, we can take as a representative class the set of all unit vectors, in which case we identify all elements of that set which are related by left multiplication by unit quaternions. Recalling that the unit quaternions form a group isomorphic to $SU(2)$, we may write $S^{4n+3} \xrightarrow{S^3} P^n$.

For low momenta, the gauge field is entirely determined by the gradients of the scalar field:

$$A_\mu^A = -\frac{1}{2g} \frac{(Q i \sigma^A, \overset{\leftrightarrow}{\partial}_\mu Q)}{(Q, Q)} \quad (9)$$

while the scalar field keeps to its vacuum manifold. Thus the effective lagrangian at low momentum is

$$\mathcal{L}_{HP} = \frac{1}{2} \sum_a \left[\text{tr}(\partial_\mu q_a^\dagger \partial^\mu q_a) - \text{tr}(\partial_\mu q_a^\dagger q_a) \text{tr}(q_a^\dagger \partial^\mu q_a) / \sum_a \text{tr}(q_a^\dagger q_a) \right] \quad (10)$$

which is that of the P^n σ -model with the Fubini-Study metric [13, 14].

The use of quaternions may seem like unnecessary obfuscation. However, there is a representation of this theory in terms of $n+1$ complex doublets ϕ_a , since each quaternion

may be written

$$q_a \equiv (\phi_a \ i\sigma^2 \phi_a^*) \quad (11)$$

by which we mean that each entry in the brackets is a column in the matrix representing the quaternion. Thus we are in essence considering a model which $n + 1$ copies of the scalar sector of the Electroweak theory, equipped with a $Sp(n + 1)$ global symmetry, in the limit $\theta_W = 0$.

A case of particular interest is the smallest member of the new class, where $n = 1$, and so we will examine it in detail. Here we have the symmetry group $Sp(2)_g \times Sp(1)_l \simeq Spin(5)_g \times SU(2)_l$ [19]. The representation of the global group in which the scalar field lies is the fundamental of $Sp(2)$ i.e. a (pseudo) real **4**, and that of the local group is the fundamental of $Sp(1)$. Thus $\Phi \equiv (\phi_1, \phi_2)^T$ transforms as a $(\mathbf{4}, \mathbf{2})$ of $Sp(2)_g \times Sp(1)_l$.

We now analyze how the representations decompose in the symmetry breaking. First, note that $Sp(1) \times Sp(1) \subset Sp(2)$ (this is just the $Spin(4) \subset Spin(5)$ subgroup chain). Under this subgroup, the **4** of $Sp(2)$ decomposes as $\mathbf{4} \rightarrow (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1})$. Thus, when one of these components acquires a vacuum expectation value, one of the $SU(2)$'s will be broken while the other remains untouched. However, at the same time, the *local* $SU(2)_l$ is also broken via a doublet. Thus just as in chiral $SU(2) \times SU(2)$ models, the generators of both of the broken $SU(2)$'s can be combined so as to form an unbroken *global* $SU(2)'_g$. The symmetry breaking pattern is then just $Sp(2)_g \times SU(2)_l \rightarrow Sp(1)_g \times SU(2)'_g$. We see from this that the vacuum manifold is of dimension $(10 + 3) - (3 + 3) = 7$. Furthermore, one can think of $Sp(2) \simeq Spin(5)$ as the product $S^7 \times S^3$, while the unbroken subgroup is just $S^3 \times S^3$. Thus, by “cancelling” the S^3 's in forming the coset, we see that the vacuum manifold should be S^7 . This is, of course, obvious from the Higgs potential. The only renormalizable $Sp(2)_g \times SU(2)_l$ invariant potential is $V(\Phi) = \frac{1}{2}\lambda(\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 - \eta^2)^2$, which contains eight real fields. It is clear from the form of $V(\Phi)$ that the vacuum

manifold is S^7 . However, just as in the $n = 1$ EAH model, the broken gauge symmetries $SU(2)_l \simeq S^3$ act non-trivially on this S^7 , yielding the Hopf bundle $S^7 \xrightarrow{S^3} S^4$.

Thus the low momentum effective σ -model has a target space S^4 , and it is no surprise that the Lagrangian can be rewritten in terms of an $O(5)/O(4)$ σ -model. This parallels the equivalence between P^1 and S^2 [20, 9]. We can make use of the isomorphism between $Sp(2)$ and $Spin(5)$, and utilize the 5 gamma matrices Γ^m to construct 5 real fields φ^m as follows:

$$\varphi^m = (Q, \Gamma^m Q) / \eta \quad (12)$$

Recalling the relation

$$\sum_m \Gamma_{\mu\nu}^m \Gamma_{\rho\sigma}^m = \frac{4}{3} \delta_{\mu\sigma} \delta_{\nu\rho} - \frac{1}{3} \delta_{\mu\nu} \delta_{\rho\sigma} \quad (13)$$

we find that in terms of the 5 real fields the σ -model Lagrangian is

$$\mathcal{L}_{O(5)} = \frac{3}{8} \sum_m \partial_\mu \varphi^m \partial^\mu \varphi^m \quad (14)$$

(supplemented by the constraint $\sum_m (\varphi^m)^2 = \eta^2$).

Let us now consider the finite energy, or more appropriately, finite Euclidean action, topological defects in this class of models. In D dimensions the action functional is

$$S = \int d^D x \left[\frac{1}{2} (F_{\mu\nu}, F_{\mu\nu}) + (D_\mu Q, D_\mu Q) + V(Q) \right] \quad (15)$$

where μ runs from 1 to D . Each term on the right hand side must vanish separately faster than $|x|^{-D}$ for a finite action solution, which means that not only must Q lie in M at infinity, but also that it be a gauge transform of the vacuum:

$$Q = \Omega(\hat{x}_\mu) Q_0 \quad (16)$$

where $\Omega(\hat{x}_\mu)$ is an $Sp(1)$ matrix. Thus at infinity, Q must lie entirely in one of the gauge orbits, which are 3-spheres. The finite action configurations on this theory are therefore classified by $\pi_{D-1}(S^3)$, not by the homotopy groups of the vacuum manifold,

S^{4n+3} . Of particular interest is the 4 dimensional case, for which the finite action field configurations fall into homotopically inequivalent classes labelled by $\pi_3(S^3) = \mathbb{Z}$. The non-trivial elements correspond to instantons [20]. The instanton *solutions* to the field equations following from this action exist only for $\eta = 0$, when the conformal symmetry of the pure Yang-Mills theory is re-established [21]. Nonetheless, instanton *configurations* are important in the quantum theory, where the path integral ensures that classically disallowed configurations contribute to physical quantities. In the $SU(2)_I$ theory with one scalar doublet, the instantons that contribute the most are those that are stable to all perturbations except dilatations, which near the origin have the form [22]

$$q_1 = \frac{e_\mu x_\mu}{(x^2 + \rho^2)^{\frac{1}{2}}} \quad gA_\mu^A = \frac{\eta_{\mu\nu}^A x_\nu}{(x^2 + \rho^2)^{\frac{1}{2}}} \quad (17)$$

Here, $\eta_{\mu\nu}^A = -i\text{tr}(e_\mu^\dagger \sigma^A e_\nu)$ [21]. Note that the Higgs field has to vanish at the centre of the instanton, because on the 3-sphere at infinity (S_∞^3) it wraps exactly once around its vacuum manifold, which is topologically S^3 . However, in our semilocal extension to this $SU(2)$ model, there is no need for the other $SU(2)$ doublets to vanish at the origin. Indeed, for very large instantons we know that the theory is approximately an P^n σ -model, and thus the field will stick close to its target space everywhere. Since the scalar field must be a gauge transformation at infinity, it maps S_∞^3 to a single point in P^n , while over the rest of the spacetime ⁴ it completes a non-contractible 4-sphere in P^n . The case $n = 1$ is simple to deal with, for there $P^1 \simeq S^1$, and as a result we have the luxury of being able to construct the instanton configurations in an obvious way out of the real fields φ^m . Large instantons constrained to have scale size ρ will take the form

$$\varphi^\mu = \sin \psi(x/\rho) \hat{x}_\mu \quad \varphi^5 = \cos \psi(x/\rho) \quad (18)$$

with $\psi(0) = 0$ and $\psi(\pi) = \pi$. The exact form of $\psi(x/\rho)$ will depend on how the instantons are constrained [22].

Similar remarks apply to the sphalerons [23] in the model. In the construction of the sphaleron of the 1-doublet theory, the 2-sphere at spatial infinity covers the entire vacuum manifold as the loop parameter τ varies between 0 and 1. The continuity of the field then implies that somewhere in $^3 \times [0, 1]$ the field must vanish. When there are n other scalar doublets, there is no topological argument forcing the other fields to vanish if their $SU(2)_l$ orientation remains fixed. Whether or not the scalar fields actually do vanish at the centre of the sphaleron is a dynamical question, as for the semilocal string.

To summarise the results so far: we have constructed another class of semilocal theories realising the fibrations $S^{4n+3} \xrightarrow{S^3} P^n$, the first member of this class being the higher Hopf map $S^7 \xrightarrow{S^3} S^4$. Emboldened by this success, one might be tempted to try and realise the highest Hopf map $S^{15} \xrightarrow{S^7} S^8$, and its companion octonionic projective space $F_4/Spin(9)$. However, we have been unable to do this in the current framework of a spontaneously broken theory with gauge and global symmetries, for the canonical semilocal construction fails for this Hopf fibration. The natural thing to do is to again duplicate the scalar field content. Thus we would have 4 quaternionic fields, or equivalently 2 fields Q_1 and Q_2 in the **4** of $Sp(2)$. If we gauge this $Sp(2)$, then the breaking $Sp(2) \rightarrow Sp(1)$ gives us the required gauged S^7 . In order to make the full vacuum manifold S^{15} , we must use the $O(16)$ symmetric potential $\lambda[(Q_1, Q_1) + (Q_2, Q_2) - \eta^2]^2$. However, this does not fix the relative orientation of Q_1 and Q_2 , which is necessary to ensure an unbroken gauged $Sp(1)$. This would entail introducing a term $\lambda'(Q_1, Q_2)(Q_2, Q_1)$, which reduces the dimension of the vacuum manifold. We have been unable to prove that a semilocal theory realising this fibration does not exist, but we have not succeeded in finding one.

However, we can generalise the existing semilocal theories in other directions. For example, P^n is one member of the family of complex Grassmanian manifolds $G(n, m) = SU(n+m)/S[U(n) \times U(m)]$ [19]. This suggests constructing a theory in which a $U(m)$ gauge symmetry is completely broken by $n+m$ scalars in the fundamental of $U(m)$. Let

us denote them by ϕ_a , where $a = 1, \dots, n + m$. Drawing on the Grassmanian σ -model construction of Brézin et al [15], it is easy to construct the correct potential that gives us the required symmetry breaking:

$$V = \frac{1}{2}\lambda(\bar{\phi}^a\phi_a - m\eta^2)^2 + \frac{1}{2}\lambda'(\bar{\phi}^a\phi_b - \eta^2\delta_b^a)^2 \quad (19)$$

When $\lambda' > 0$ this ensures that the vacuum solution satisfies

$$\bar{\phi}^a\phi_b = \eta^2 P_b^a \quad (20)$$

where P_b^a is a rank m projector onto the broken part of the global $SU(n + m)$ symmetry. We can always choose a basis in which $\phi_{ia} = \delta_{ia}$ for $i, a = 1, \dots, m$, so that it is clear that the local $U(m)$ mixes with a $U(m)$ subgroup of the global symmetries to form an unbroken global subgroup. Thus the symmetry breaking pattern is

$$SU(n + m)_g \times U(m)_l \rightarrow SU(n)_g \times U(m)_g \quad (21)$$

The full vacuum manifold M is the space of m -frames in \mathbb{R}^{n+m} , otherwise known as the Stiefel manifold $V_{n+m,m}()$, which is isomorphic to $SU(n + m)/SU(n)$. This space is fibred by the $U(m)$ gauge orbits, which must be factored out to obtain the low energy σ -model target space M_g . Thus we realize the bundle

$$V_{n+m,m}() \xrightarrow{U(m)} G(n, m). \quad (22)$$

Grassman manifolds exist over the other commutative division algebras and [19], and they can be constructed in a semilocal context by replacing the complex fields of the preceding discussion by real or quaternionic ones respectively. This has the effect of changing the unitary symmetries into orthogonal or symplectic ones, and the symmetry breaking patterns become

$$SO(n + m)_g \times SO(m)_l \rightarrow SO(n)_g \times SO(m)_g$$

$$Sp(n + m)_g \times Sp(m)_l \rightarrow Sp(n)_g \times Sp(m)_g$$

Preskill [10] has constructed an example of a semilocal theory which in a sense mixes the real and complex cases. One can choose to gauge only an $SO(m)$ subgroup of the $U(m)$ symmetry in the $G(n, m)$ model, producing the breaking

$$U(n + m)_g \times SO(m)_l \rightarrow U(n)_g \times SO(m)_g \quad (23)$$

When $n = 0$ and $m = 3$ we obtain Preskill's model. All the models over and have topologically non-trivial vortex configurations, since $\pi_1(G_l)$ is non-trivial for $G_l \simeq SO(m)$ and $U(m)$. When the gauge group is $SO(2)$ or $U(m)$, the vortices are labelled by an integer, because of the broken $U(1)$ factor. When the gauge group is $SO(m)$ ($m > 2$), the first homotopy group is \mathbb{Z}_2 .

Lastly, we point out that there are quaternionic projective spaces which are not Grassmanian. They are finite in number, and involve the exceptional groups [13]:

$$\begin{aligned} G_2/SU(2) \times Sp(1) \quad F_4/Sp(3) \times Sp(1) \quad E_6/SU(6) \times Sp(1) \\ E_7/Spin(12) \times Sp(1) \quad E_8/E_7 \times Sp(1) \end{aligned}$$

It would be interesting to construct semilocal models which realise these manifolds as their σ -model target spaces.

To conclude, we have constructed several more classes of semilocal model, all of whose low energy σ -model target manifolds are Grassman manifolds. Any GUT with continuous global symmetries must confront the issue of semilocality, so these investigations are expected to have some importance. Their value lies particularly in the sphere of the cosmological predictions of the theory: for example, a semilocal theory may not have the stable cosmic strings that the gauge sector symmetry breaking may predict; or the strings may terminate on global monopoles [9, 11]. We hope to return to the subject of semilocal GUTs in a future publication.

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